NONSTATIONARY POTENTIAL FLOW OF A POLYTROPIC GAS WITH DEGENERATE HODOGRAPH

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We shall consider the nonstationary potential flow of a gas, assuming a polytropic equation of state. For this case, the velocity components u_1, u_2, u_3 in Cartesian coordinates x_1, x_2, x_3, and the square of the velocity of sound θ satisfy the equations

$$\frac{\partial u_i}{\partial t} + \sum_k u_k \frac{\partial u_i}{\partial x_k} + \varkappa \frac{\partial \theta}{\partial x_i} = 0 \quad (i = 1, 2, 3) \quad \text{(Euler's equations)} \tag{1}$$

$$\varkappa \frac{\partial \theta}{\partial t} + \theta \sum_k \frac{\partial u_k}{\partial x_k} + \varkappa \sum_k u_k \frac{\partial \theta}{\partial x_k} = 0 \quad \text{(continuity equation)}$$

rot $\mathbf{u} = 0$ (potential condition).

Here $\kappa = 1/(\gamma - 1)$ and γ is the adiabatic index. The solution of the system of equations (1) gives a travelling wave of rank r if the rank of the matrix A

$$A = \left| \begin{array}{cccc} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \frac{\partial u_1}{\partial t} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial \theta}{\partial x_1} & \frac{\partial \theta}{\partial x_2} & \frac{\partial \theta}{\partial x_3} & \frac{\partial \theta}{\partial t} \end{array} \right| \tag{2}$$

is equal to r, for a given solution [1]. Nikolsky, with the help of a "ranking function" Δ which he introduced, investigated functions of the second rank,

$$\nabla(u_1, u_2) = \sum_k u_k x_k - \varphi \tag{3}$$

for the case of potential, stationary flows (here ϕ is the velocity potential). He obtained the differential equations for the functions Λ and ψ, assuming that $u_3 = \psi(u_1, u_2)$. Ryzhov [2] investigated double waves in the case of potential nonstationary flow. In reference [1] double waves in two dimensions were studied, without assuming the flow to be potential.

In the present note we investigate triple waves (waves of rank 3) for the case of three-dimensional, potential, nonstationary flow, by introducing the "ranking function" (u_1, u_2, u_3, t).

We shall first consider the general case, for which $\theta = \theta(u_1, u_2, u_3)$ and u_1, u_2, u_3 are functionally independent; the case $u_3 = \psi(u_1, u_2)$ will be considered later.

Since the flow is potential, we have Cauchy's integral

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(u_1^2 + u_2^2 + u_3^2) + \varkappa\theta = F(t) \tag{4}$$

where $F(t)$ is an arbitrary function of time, and

$$\frac{\partial \varphi}{\partial x_i} = u_i \qquad (i=1, 2, 3) \tag{5}$$

Next we introduce the function Δ defined by

$$\nabla = \sum_k x_k u_k - \varkappa t\theta - \varphi \tag{6}$$

and write its total differential

$$d\nabla = \sum_k (x_k - \varkappa t\theta_k)\, du_k - \left(\frac{\partial \varphi}{\partial t} + \varkappa\theta\right) dt, \qquad \theta_k = \frac{\partial \theta}{\partial u_k} \tag{7}$$

Thus Δ is a function of u_1, u_2, u_3, and t, its partial derivatives being as follows:

$$\frac{\partial \nabla}{\partial u_i} = x_i - \varkappa t\theta_i, \qquad \frac{\partial \nabla}{\partial t} = -\frac{\partial \varphi}{\partial t} - \varkappa\theta \tag{8}$$

Making use of equation (4), the second equation of (8) may be written in the form

$$\frac{\partial \nabla}{\partial t} = \frac{1}{2}(u_1^2 + u_2^2 + u_3^2) - F(t) \tag{9}$$

Integrating (9) with respect to t, we obtain

$$\nabla = \frac{1}{2}(u_1^2 + u_2^2 + u_3^2)\, t + \Phi(u_1, u_2, u_3) + F^\circ(t), \quad F^\circ(t) = -\int F(t)\, dt \tag{10}$$

where $\Phi(u_1, u_2, u_3)$ is some function, undetermined so far.

We put $\Phi(u_1, u_2, u_3)$ in the form

$$\Phi(u_1, u_2, u_3) = \frac{1}{2}(u_1^2 + u_2^2 + u_3^2) + \varkappa\Pi(u_1, u_2, u_3) \tag{11}$$

and differentiate (10) with respect to u_i. Then the first equation of (8) may be presented in the form

$$x_i = \varkappa\Pi_i + u_i + t(\varkappa\theta_i + u_i), \qquad \Pi_i = \frac{\partial \Pi}{\partial u_i} \tag{12}$$

Euler's equation of motion will be satisfied if Cauchy's integral holds, and it is necessary only to satisfy the continuity equation. Using the equations

$$\frac{\partial\theta}{\partial t} = \sum_k \theta_k \frac{\partial u_k}{\partial t}, \qquad \frac{\partial\theta}{\partial x_i} = \sum_k \theta_k \frac{\partial u_k}{\partial x_i} \tag{13}$$

and substituting $\partial u_k/\partial t$ from Euler's equations, we put the continuity equation in the form

$$\sum_{ik} A_{ik} \frac{\partial u_i}{\partial x_k} = 0 \qquad (A_{ik} = \delta_{ik}\theta - \varkappa^2\theta_i\theta_k) \qquad (i, k=1, 2, 3) \tag{14}$$

Let

$$\Delta = \frac{D(x_1, x_2, x_3)}{D(u_1, u_2, u_3)} \neq 0 \tag{15}$$

in some region of x_1, x_2, x_3, t. Then, carrying out the hodograph transformation between the variables u_1, u_2, u_3 and x_1, x_2, x_3, in equation (14), we will have, for fixed t,

$$\sum_{ik} A_{ik} L_{ik} = 0 \quad (i, k=1, 2, 3), \qquad L_{ik} = (-1)^{i+k} \begin{vmatrix} \dfrac{\partial x_m}{\partial u_p} & \dfrac{\partial x_n}{\partial u_p} \\ \dfrac{\partial x_m}{\partial u_q} & \dfrac{\partial x_n}{\partial u_q} \end{vmatrix} \begin{array}{l} (m, n \neq k,\ m<n) \\ \\ (p, q \neq i,\ p<q) \end{array} \tag{16}$$

From (12) we obtain

$$\frac{\partial x_i}{\partial u_k} = \varkappa\Pi_{ik} + \delta_{ik} + t(\varkappa\theta_{ik} + \delta_{ik}), \ \Pi_{ik} = \frac{\partial^2 \Pi}{\partial u_i \partial u_k}, \quad \theta_{ik} = \frac{\partial^2\theta}{\partial u_i \partial u_k} \tag{17}$$

where δ_{ik} is the Kronecker symbol.

With (17), equations (16) may be put in the form

$$\Gamma_0 + \Gamma_1 t + \Gamma_2 t^2 = 0 \tag{18}$$

where Γ_0, Γ_1, Γ_2 are functions only of u_1, u_2, u_3.

Since equation (18) is valid for arbitrary values of t, we conclude that

$$\Gamma_i = 0 \qquad (i=0, 1, 2) \tag{19}$$

in which the expression for Γ_i is as follows:

$$\Gamma_j = \sum_{ik} A_{ik} \overset{\bullet}{L}{}_{ik}^{j} \qquad (j=0, 1, 2)$$

where

$$L_{ik}{}^{0} = (-1)^{i+k} \begin{vmatrix} \varkappa\Pi_{mp} + \delta_{mp} & \varkappa\Pi_{np} + \delta_{np} \\ \varkappa\Pi_{mq} + \delta_{mq} & \varkappa\Pi_{nq} + \delta_{nq} \end{vmatrix}$$

$$L_{ik}{}^{1} = (-1)^{i+k} \left\{ \begin{vmatrix} \varkappa\Pi_{mp} + \delta_{mp} & \varkappa\Pi_{np} + \delta_{np} \\ \varkappa\theta_{mq} + \delta_{mq} & \varkappa\theta_{nq} + \delta_{nq} \end{vmatrix} + \begin{vmatrix} \varkappa\theta_{mp} + \delta_{np} & \varkappa\theta_{np} + \delta_{np} \\ \varkappa\Pi_{mq} + \delta_{mq} & \varkappa\Pi_{nq} + \delta_{nq} \end{vmatrix} \right\}$$

$$L_{ik}{}^{2} = (-1)^{i+k} \begin{vmatrix} \varkappa\theta_{mp} + \delta_{mp} & \varkappa\theta_{np} + \delta_{np} \\ \varkappa\theta_{mq} + \delta_{mq} & \varkappa\theta_{nq} + \delta_{nq} \end{vmatrix}$$

and in all relations we have $m,\ n \neq k,\ m < n;\ p,\ q \neq i,\ p < q$.

The equation $\Gamma_2 = 0$ is a nonlinear, second order partial differential equation for the function θ. It may be posed as the Cauchy problem, or, analogously to the problem posed in reference [1] as a Goursat problem with two arbitrary functions of two variables.

Without posing any definite gasdynamic problems and without investigating, in the present note, questions of uniqueness of solutions, we note that, after the function θ is found, the system of equations $\Gamma_0 = 0$ and $\Gamma_1 = 0$ (in which the function Π appears) is compatible, and has, for instance, the solutions

$$\Pi = \theta + \sum_k c_k u_k + C \qquad (c_k = \text{const},\ C = \text{const}) \tag{20}$$

In the solution of a definite gasdynamic problem, it is necessary, after determining the function θ, to find the function Π which satisfies the two equations $\Gamma_0 = 0$ and $\Gamma_1 = 0$, and the particular conditions of the problem, in order to obtain a unique solution.

After determination of the functions Π and θ, the flow in the x_1, x_2, x_3, t plane is found from equations (8).

We note that a completely analogous application of the method in the two-dimensional case leads to two differential equations for the functions Φ and θ, which are identical with the equations obtained for that case in reference [1].

Next we examine the functional dependence $u_3 = \psi(u_1, u_2)$. We introduce, as before, the function Δ, but with a more restricted dependence on t:

$$\nabla = \sum_k u_k x_k - \varphi \tag{21}$$

Taking the total differential, we find that Δ is a function of u_1, u_2, t

$$\frac{\partial\nabla}{\partial u_i} = x_i + \psi_i x_3, \qquad \frac{\partial\nabla}{\partial t} = -\frac{\partial\varphi}{\partial t}, \qquad \psi_i = \frac{\partial\psi}{\partial u_i} \quad (i=1,2) \tag{22}$$

Using relation (4) and also the relations

$$-\frac{\partial u_i}{\partial t} = \frac{\partial^2 \nabla}{\partial t\,\partial x_i} \tag{23}$$

which follow from the second equation in (22), we calculate the derivatives $\partial\theta/\partial x_i$, $\partial\theta/\partial t$, $\partial u_3/\partial x_i$ and put the expressions for them in the continuity equation. This then takes the form

$$R_0 + R_1\frac{\partial u_1}{\partial x_1} + R_2\frac{\partial u_1}{\partial x_2} + R_3\frac{\partial u_2}{\partial x_1} + R_4\frac{\partial u_2}{\partial x_2} = 0 \tag{24}$$

Here

$$R_0 = F'(t) + \frac{\partial^2\nabla}{\partial t^2}, \quad R_1 = -\left(\frac{\partial^2\nabla}{\partial t\,\partial u_1} - \psi\psi_1 - u_1\right)^2 + \theta(1+\psi_1^2) \tag{25}$$

$$R_2,\ R_3 = -2\left(\frac{\partial^2\nabla}{\partial t\,\partial u_1} - u_1 - \psi\psi_1\right)\left(\frac{\partial^2\nabla}{\partial t\,\partial u_2} - u_2 - \psi\psi_2\right) + 2\theta\psi_1\psi_2$$

$$R_4 = -\left(\frac{\partial^2\nabla}{\partial t\,\partial u_2} - u_2 - \psi\psi_2\right)^2 + \theta(1+\psi_2^2)$$

$$0 = \frac{1}{\varkappa}\left[F(t) + \frac{\partial\nabla}{\partial t} - \frac{1}{2}(u_1^2 + u_2^2 + \psi^2)\right]$$

In equation (24) we carry out the hodograph transformation for the pairs of variables u_1, u_2 and x_1, x_2. Let

$$\Delta = \frac{D(x_1, x_2)}{D(u_1, u_2)} \neq 0$$

in some region. We exclude the case $\Delta = 0$. Differentiating the first equality in (22) with respect to u_1, u_2, for fixed x_3, t, we find the expressions for $\partial x_i/\partial u_k$ $(i, k = 1, 2)$. Then, after carrying out the transformations, equations (24) may be put in the form

$$T_0 + T_1 x_3 + T_2 x_3^2 = 0 \tag{26}$$

where T_0, T_1, T_2 are functions of u_1, u_2, t.

Since x_3 is arbitrary, it is necessary that $T_i = 0$, $i = 0, 1, 2$, the expressions for T_i then become as follows:

$$T_1 = -R_0\left(\frac{\partial^2\nabla}{\partial u_1^2}\frac{\partial^2\psi}{\partial u_2^2} + \frac{\partial^2\nabla}{\partial u_2^2}\frac{\partial^2\psi}{\partial u_1^2} - 2\frac{\partial^2\nabla}{\partial u_1\partial u_2}\frac{\partial^2\psi}{\partial u_1\partial u_2}\right) - R_1\frac{\partial^2\psi}{\partial u_2^2} + (R_2+R_3)\frac{\partial^2\psi}{\partial u_1\partial u_2} - R_4\frac{\partial^2\psi}{\partial u_1^2} = 0 \tag{27}$$

$$T_0 = R_0\left[\frac{\partial^2\nabla}{\partial u_1^2}\frac{\partial^2\nabla}{\partial u_2^2} - \left(\frac{\partial^2\nabla}{\partial u_1\partial u_2}\right)^2\right] + R_1\frac{\partial^2\nabla}{\partial u_2^2} - (R_2+R_3)\frac{\partial^2\nabla}{\partial u_1\partial u_2} + R_4\frac{\partial^2\nabla}{\partial u_1^2} = 0$$

$$T_2 = R_0\left[\frac{\partial^2\psi}{\partial u_1^2}\frac{\partial^2\psi}{\partial u_2^2} - \left(\frac{\partial^2\psi}{\partial u_1\partial u_2}\right)^2\right] = 0 \tag{28}$$

Equation (28) allows two possibilities. Let us consider the first case

$$R_0 = F'(t) + \frac{\partial^2\nabla}{\partial t^2} = 0$$

Hence

$$F(t) + \frac{\partial \nabla}{\partial t} = \Lambda(u_1, u_2), \qquad \nabla = \Lambda(u_1, u_2)\, t + \chi(u_1, u_2) - \int F(t)\, dt \tag{29}$$

where Λ and χ are certain functions.

In this case equations (27) will give three third order equations for ψ, Λ, and χ, which are the same as the system obtained in reference [2], and which describe double waves.

Let us consider the second case:

$$\frac{\partial^2\psi}{\partial u_1^2}\frac{\partial^2\psi}{\partial u_2^2} - \left(\frac{\partial^2\psi}{\partial u_1 \partial u_2}\right)^2 = 0 \tag{30}$$

Equation (30) is the equation of developed surfaces, if cylindrical surfaces of the form $f(u_1, u_2) = \text{const}$ are excluded. In this case $\Delta(u_1, u_2, t)$ has to satisfy equations (27), and it is necessary, generally speaking, to investigate their compatibility for a chosen ψ.

We have an example, for which these two equations prove to be compatible and new flows are obtained. Specifically, consider a flow with

$$\psi = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 \qquad (\alpha_i = \text{const})$$

The equation $T_1 = 0$ is automatically satisfied for such a flow. For $\Delta(u_1, u_2, t)$ there is left one equation, $T_0 = 0$, with $\psi_1 = \alpha_1$, $\psi_2 = \alpha_2$ as its coefficients.

We also note that all the flows investigated have straight characteristics in the x_1, x_2, x_3, t plane, as follows from equations (8) and (22).

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